数理解析研究所講究録
1122 巻 2000 年 1-25

Introduction to Resolution of Singularities

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We will present an elementary proof of a canonical resolution of singularities in characteristic zero (at least in the hypersurface case) including detailed examples illustrating some (elementary, but important) applications and the constructive features of the "local to global" argument. The proof is by introduction of a discrete local invariant whose maximum locus determines a smooth centre of blowing up, leading to desingularization.

Lecture 1

Blow ups, Desingularization Theorems and examples.

Lecture 2

Proof of Weak Desingularization (in all details). From local to global: local properties of an invariant that imply a global desingularization.

Lecture 3

Constructive definition of the invariant for desingularization and an example illustrating the construction.

TABLE OF CONTENT

PART 1. BRIEF HISTORY OF DESINGULARIZATION AND THE MAIN FEATURES (OF MY WORK WITH BIERSTONE)

History.

The problem of resolution of singularities appeared in the middle of the nineteenth century, altough in the 1-dimensional case it existed earlier in the guise of finding **good parametrization** of curves. At first, the problem was only considered for $\mathbb{R}$ and $\mathbb{C}$.

dim $X = 1$: Puiseux, Kronecker, Halphen, XIX century, I. Newton, XVII century.
RIEMANN, XIX century.

dim $X = 2$, char $k = 0$: Beppo Levi 1897, Chisini 1921, Albanese 1924, R.J. Walker – Jung 1935, Zariski 1939, 1942.

Local Uniformization Theorem, any dimension, char $k = 0$: Zariski, 1940.

dim $X = 3$, char $k = 0$: Zariski, 1944.

DESINGULARIZATION HISTORY (very brief):

A. <u>By BLOW UPS</u>:

NEWTON ... ZARISKII ... HIRONAKA
in char $k=0$ ('60 - '90)

BLOW UPS INTRODUCED BY:
MAX NOETHER ... ZARISKII

B. <u>By projections, ALTERATIONS</u>:

ALBANESE ... de JONG
char $k \neq 0$ (from '96 - '97)

DRAWBACK: POOR CONTROL OF THE DESINGULARIZATION MAP.

C. <u>By NASH BLOW UPS</u>:

in dim $=2$
NASH ... HIRONAKA (SPIVAKOVSKII)

FROM AMS FEATURED REVIEW by J. Lipman:

Hironaka's theorem on the existence of resolutions of singularities for any algebraic or analytic variety V over a field of characteristic zero is an outstanding achievement of twentieth-century mathematics, by virtue of the depth both of its proof and of its applications. ...The history of the problem of existence of resolutions goes back more than a century ... . And the history is ongoing ... novel approaches to global desingularization have just been developed by A. J. de Jong et al ..., leading to a new generation of short, but non-constructive, proofs. Hironaka's proof is lengthy, difficult, and non-constructive. Influential as the proof has been, few people can have checked it through entirely, even after some subsequent enhancements of the machinery ... . Simplified, more algorithmic proofs are important not only for imparting better understanding of what is really involved in this great theorem, but also for their potential value in unearthing basic features of singularities and their classification. The challenge of finding more straightforward algorithmic approaches was taken up by Zariski, Abhyankar, and others, and successfully met only in the past decade by Bierstone, Milman, and Villamayor.

THE MAIN FEATURES (BIERSTONE-Milman, '88-'97)

1. CANONICAL DESINGULARIZATION

2. LOCAL PROPERTIES (OF AN INVARIANT)

$\Rightarrow$ GLOBAL RESOLUTION.

3. IN THE HYPERSURFACE CASE INDUCTION DOES NOT INVOLVE PASSING TO CODIMENSION > 1.

IN CHARACTERISTIC ZERO (THOUGH THE REDUCTION TO THE HYPERSURFACE CASE WORKS EVEN IN NONEZERO CHARACT.)

4. APPLIES (IN THE LANGUAGE OF LOCALLY RINGED SPACES) TO spaces S THAT ARE LOCALLY $S \hookrightarrow \mathcal{U}$ SMOOTH ETALE "COORDINATE CHARTS", WHICH WE DEFINE BELOW. (WE ALSO REQUIRE THAT $\mathcal{O}_S$ - COHERENT, $|S|$ - locally

NOETHERIAN AND ONE MORE TECHNICAL PROPERTY – "PRIVILEDGED NBD. PROPERTY".)

ELEMENTS OF $\mathcal{O}(\mathcal{U})$ CALLED REGULAR FUNCTIONS, E.G. POLYNOMIALS, ANALYTIC, QUASIANALYTIC, ...

DEFINITION $\mathcal{U}$ is a smooth etale "COORDINATE CHART" IFF EXIST $x_1, \ldots, x_n \in \mathcal{O}(\mathcal{U})$, which we CALL "COORDINATES", AND a ("TAYLOR") homomorphism $T : \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}[[X]]$ into the RING OF FORMAL POWER SERIES EXPANSIONS IN $X = (X_1, \ldots, X_n)$ SUCH THAT LETTING

$\sum_{\alpha} f_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \overset{def}{=} Tf$ WE HAVE:

(1.) $f_0 = f$; (2.) $T x_j = x_j + X_j \ \forall j$;

(3.) DENOTE $D_{\alpha} f \overset{def.}{=} f_{\alpha}$ AND FOR A POWER SERIES EXPANSION $F(X)$ LET $D_{\alpha} F(X)$ BE DEFINED BY $\sum_{\alpha} D_{\alpha} F(X) \cdot Y^{\alpha} = F(X+Y)$

THEN $\quad T \circ D_\alpha = D_\alpha \circ T \quad \forall \alpha$;

4. LET $T_a : \mathcal{O}_a \to \mathbb{F}_a[[X]]$ BE DEFINED BY EVALUATION OF COEFFICIENTS AT $a \in \mathcal{U}$, i.e.
$T_a f = \sum_\alpha f_\alpha(a) \cdot X^\alpha$, WHERE $\mathbb{F}_a \overset{\text{def}}{=} \mathcal{O}_a / m_a$, AND $\hat{T}_a : \hat{\mathcal{O}}_a \to \mathbb{F}_a[[X]]$ BE THE INDUCED MAP OF COMPLETIONS. THEN $\hat{T}_a$ IS AN ISOMORPHISM.

<u>An illustrating example,</u>

e.g. IN $\mathbb{k}$-algebraic CASE (OR SCHEMES OF FINITE TYPE) :

$$
\begin{tikzcd}
\mathcal{U} = \{(x,y) : P(x,y) = 0\} \arrow[r, phantom, "\subseteq"] \arrow[dr, "\pi"', "\text{etale}"] & \mathbb{A}^N_{(x,y)} \arrow[d] \\
 & \mathbb{A}^n_x
\end{tikzcd}
$$

$P = (P_1, \dots, P_{N-n})$ polynomials and π is etale, i.e.

$$\det \frac{\partial P}{\partial y} \neq 0 \quad \text{ON } \mathcal{U}$$

$x_1, \dots, x_n$ — "COORDINATES"

AND (VIA IMPLICIT DIFFERENTIATION) it is easy to define a homomorphism T

PART 2. BLOW UP, examples.

BLOW UP ALONG $I \subset \mathcal{O}_{\mathcal{U}}$

$$I = (f_1, \dots, f_k) \subseteq \mathcal{O}(\mathcal{U}), \ \mathcal{U}\text{-smooth}$$

$$\mathcal{U} - V(f_1, \dots, f_k) \xrightarrow{[f]} \mathbb{P}^{k-1}$$

$$x \longmapsto [f_1(x) : \dots : f_k(x)]$$

$$Bl_I\, \mathcal{U} := \text{Closure Graph}[f] \subseteq \mathcal{U} \times \mathbb{P}^{k-1}$$

① $Bl_I\, \mathcal{U}$ DOES NOT DEPEND ON THE CHOICE OF GENERATORS OF I

---

② IF $I = I_C$, WHERE $C \hookrightarrow M$

$Bl_C M := Bl_I M$

③ EVEN FOR SMOOTH M (UNLESS C SMOOTH $\hookrightarrow M$)

$Bl_C M$ MAY HAVE SINGULARITIES

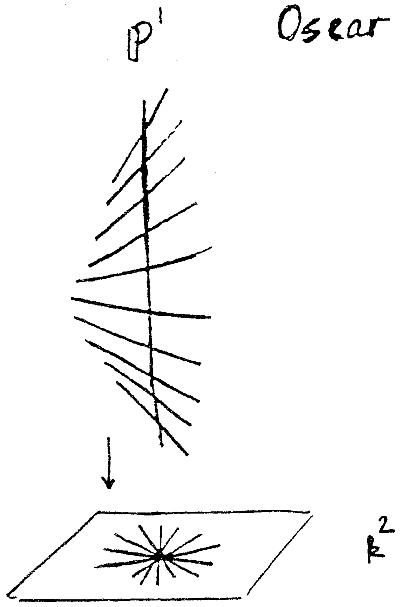
Example:

e.g. $M = \mathbb{k}^2$ $\quad C = \{0\}$ $\quad Bl_0 \mathbb{k}^2$

Blowing up.

MAX NOETHER
Oscar ZARISKI

The plane at a point:



0 is removed and replaced by $\mathbf{P}^1$ parametrizing all the lines in k^2 passing through 0. If $k = \mathbf{R}$, we get the Möbius band.

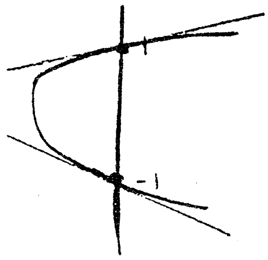
WE LIFT EVERY LINE THROUGH THE ORIGIN WITH THE SLOPE $z = \frac{y}{x}$ (y-axis HAS SLOPE ∞) TO THE HEIGHT z. Union OF ALL THESE LINES, SAY M', IS THE BLOW UP OF THE PLANE AT 0.

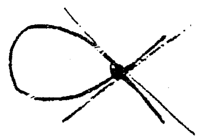
π induces an isomorphism $M' \setminus \pi^{-1}(0) \simeq k^2 \setminus \{0\}$.

Example1

Let x, y be coordinates on k^2. Algebraically, M' is glued together from two coordinate charts, with coordinates $(x, \frac{y}{x})$ and $(y, \frac{x}{y})$, respectively.

Example. Resolution of singularities of the curve C defined by $y^2 - x^2 - x^3 = 0$.

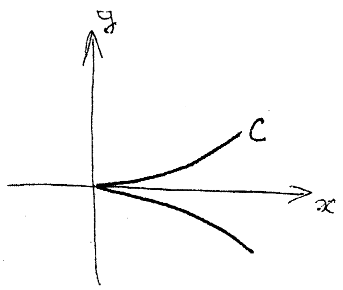




$x' = x,\ y' = \frac{y}{x}$.

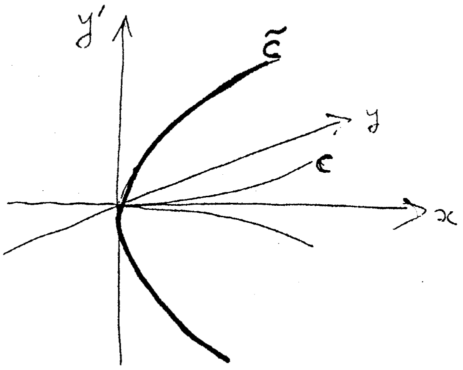
$y^2 - x^2 - x^3 = (x'y')^2 - x'^2 - x'^3 = x'^2(y'^2 - 1 - x')$.

Example 2. Resolution of singularities of the curve C defined by $y^2 - x^3 = 0$.



$x' = x,\ y' = \frac{y}{x}.$

$$y^2 - x^3 = (x'y')^2 - x'^3 = x'^2(y'^2 - x').$$



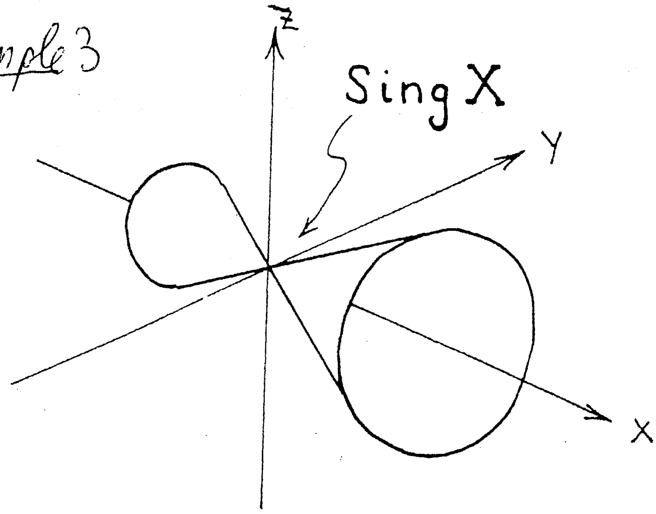
i.e. $\tilde{C}$: $x' = y'^2$

OR IN (x, y, y')-space

$x = y'^2 \quad y = y'^3$

NOT NORMAL CROSSING
(YET ...)

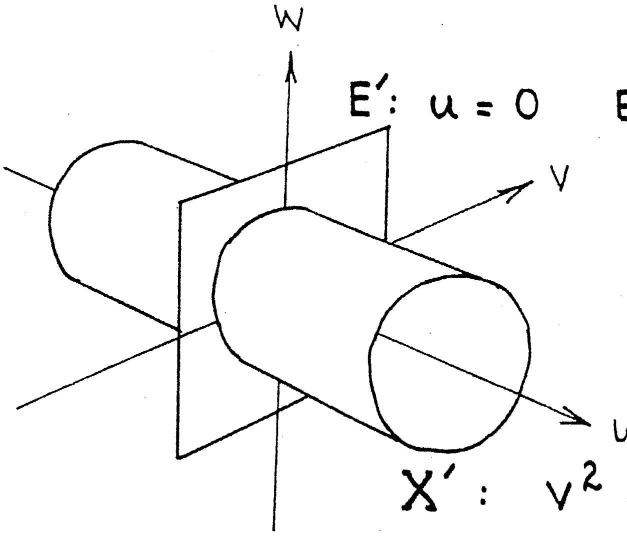
Example 3



$$X : \; x^2 - y^2 - z^2 = 0$$

$\left(x, \frac{y}{x}, \frac{z}{x}\right) = (u, v, w)$ THEN σ : $\begin{array}{l} x = u \\ y = u \cdot v \\ z = u \cdot w \end{array}$

$$\sigma^{-1}(X) : \; u^2(1 - v^2 - w^2) = 0$$



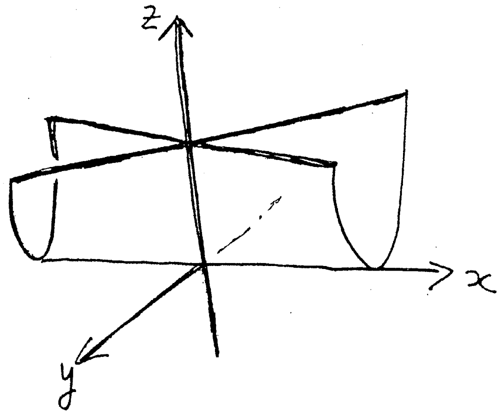
E': $u = 0$ EXCEPTIONAL HYPERSURFACE

$$X' : \; v^2 + w^2 = 1$$

STRICT TRANSFORM

Example 4

WHITNEY UMBRELLA



$X: \quad y^2 - z\cdot x^2 = 0$

BLOW UP OF O DOES NOT DESINGULARIZE:

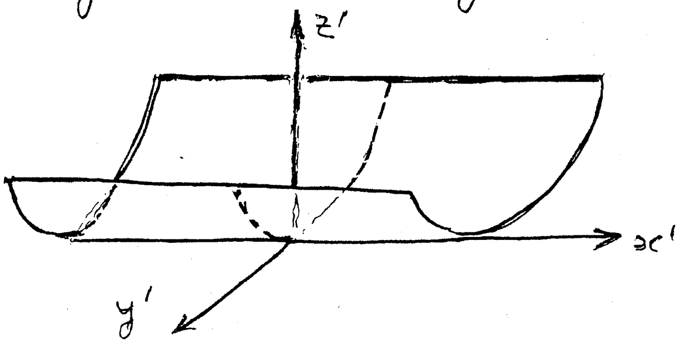
in $\left(\frac{x}{z}, \frac{y}{z}, z\right) =: (x_1, y_1, z_1)$ COORDINATES

$$y^2 - z\cdot x^2 = z_1^2\cdot y_1^2 - z_1\cdot z_1^2\cdot x_1^2 = z_1^2\cdot(y_1^2 - z_1 x_1^2)$$

BUT BLOW UP OF z-axis DOES:

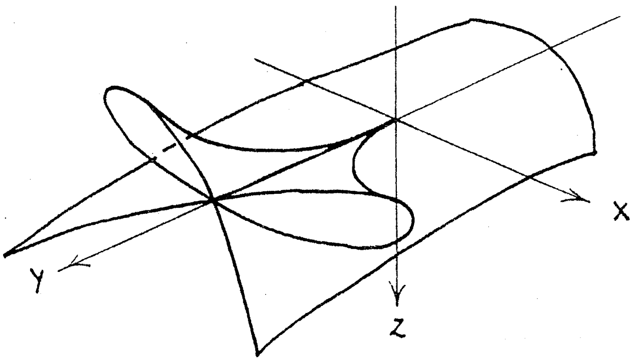
in $\left(x, \frac{y}{x}, z\right) =: (x', y', z')$ COORDINATES

$$y^2 - z\cdot x^2 = x'^2\cdot y'^2 - z'\cdot x'^2 = x'^2\cdot(y'^2 - z')$$



Example 5

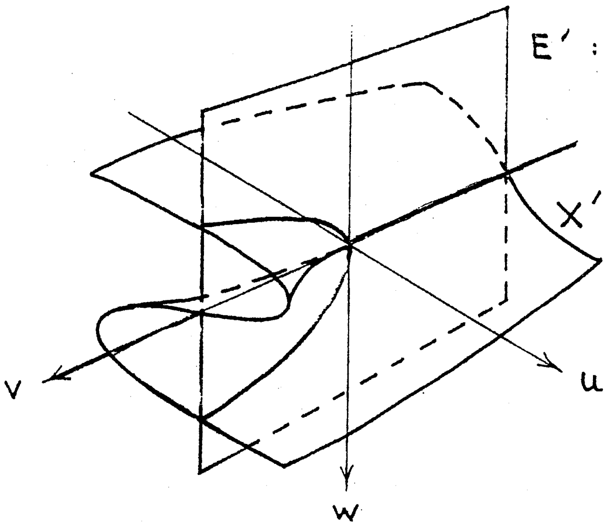
$$X : \quad z^3 - x^2yz - x^4 = 0$$



$$\Big\uparrow \sigma : \quad \begin{aligned} x &= u \\ y &= v \\ z &= uw \end{aligned}$$

BLOWING - UP
(IN LOCAL COORDINATES)

$$\sigma^{-1}(X) : \quad u^3 (w^3 - vw - u) = 0$$



$E' : \; u = 0$

$X' : \; u = w^3 - vw$

PART 3 EQUIMULTIPLE BLOW UPS.

BLOW UP ALONG SMOOTH C:

$$\begin{array}{ccc} M & \xleftarrow{\sigma} & M' = Bl_C M \\ \cup & & \cup \\ \mathcal{U} & \longleftarrow & \sigma^{-1}(\mathcal{U}) = \mathcal{U}' = Bl_{\mathcal{U}\cap C}\,\mathcal{U} \end{array}$$

coord. chart

$$\mathcal{U} \cap C = \{x_1 = \ldots = x_m = 0\}$$

$$\mathcal{U}' = \{(x,\xi) \text{ S.TH. } [x_1 : \ldots : x_m] = \xi\} \subset \mathcal{U} \times \mathbb{P}^{m-1}$$

$$\boxed{x_i \cdot \xi_j = x_j \cdot \xi_i}$$

$$\begin{array}{ccc} \mathcal{U}' & & \mathcal{U}\times\mathbb{P}^{m-1} \\ & \searrow^{\sigma} & \downarrow \pi \\ & & \mathcal{U} \end{array}$$

$$\mathcal{U}' = \bigcup_{1 \le j \le m} \mathcal{U}_j \qquad \text{each } \mathcal{U}_j = \{\xi_j \neq 0\}$$

1. each $\mathcal{U}_j$ coord. chart WITH COORD.

$$y_i = \frac{x_i}{x_j} = \frac{\xi_i}{\xi_j} \qquad \text{IF } i \neq j,\ 1 \le i \le m$$

$$y_s = x_s \qquad \text{IF } s = j \text{ or } m < s \le n$$

2. $\mathcal{U}_j = \mathcal{U}' - \{x_j = 0\}'$
3. IN coord. on $\mathcal{U}_j$ $\quad \{x_i = 0\}' = \{y_i = 0\}$ $\quad i \neq j$

BLOW UP ALONG SMOOTH C:

$$\mathcal{U} \cap C = \{x_1 = \dots = x_m = 0\}$$

$$\mathcal{U}' = \bigcup_{1 \le j \le m} \mathcal{U}_j$$

$$\begin{cases} \mathcal{U}_j \text{ coord. charts} \\ y = (y_1, \dots, y_n) \end{cases}$$

$$\sigma/_{\mathcal{U}_j}: \quad \begin{array}{ll} x_j = y_j & \\ x_i = y_j \cdot y_i & i \ne j,\ 1 \le i \le m \\ x_s = y_s & m < s \le n \end{array}$$

IF $X = V(f)$ THEN $X' = V(f')$

WHERE $(f') := (y_{EXC})^{-d} \cdot (f \circ \sigma)$

on $\mathcal{U}_j$ $\quad y_{EXC} = y_j$, $\quad \{y_{EXC} = 0\} := \sigma^{-1}(C)$

d = maximal INTEGER SUCH THAT f' is regular $= ord_C(f)$

In general, codim $X > 1$:

IF $X \subset M$ THEN $X' := \overline{\sigma^{-1}(X - C)}$

AND $I_{X'} = (f' \text{ s.th. } f \in I_X) \subset \mathcal{O}_{M'}$

THE EFFECT OF AN EQUIMULTIPLE BLOW UP:

HYPERSURFACE $\quad f(x) = 0$

$$d = \mathrm{ord}_a f$$

$f(x) =$

$$c_0(\tilde{x}) + \cdots + c_{d-1}(\tilde{x})\, x_n^{d-1} + c_d(x)\, x_n^{d}$$

$(x_1, \ldots, x_{n-1})$ $\qquad$ 0, BY COMPLETING d'TH POWER $\qquad$ ASSUME $\equiv 1$

EQUIMULTIPLE LOCUS

$$S_d := \{ x : \mathrm{ord}_x f = d \}$$

$$= \{ x : x_n = 0, \ \mathrm{ord}_{\tilde{x}}\, c_q \geq d - q \}$$

$$\min_q \mathrm{ord}_a\, c_q / (d-q)$$

EFFECT OF A BLOWING-UP σ, CENTRE $C \subset S_d$

$$C = Z_I := \{ x_n = 0, \quad x_i = 0, \ i \in I \}$$

WHERE $I \subset \{1, \dots, n-1\}$

<u>EFFECT</u> ... <u>IN CHART</u> $\mathcal{U}_i'$, $i \in I$:

$$\sigma | \mathcal{U}_i' : \quad x_j = y_i y_j \qquad j \in I \cup \{n\} \setminus \{i\}$$

$$x_j = y_j \qquad \text{OTHER } j$$

$$i \in I : \qquad f(\sigma(y)) = y_i^d f'(y),$$

$$f'(y) =$$

$$c_0'(\tilde{y}) + \cdots + c_{d-2}'(\tilde{y}) y_n^{d-2} + y_n^d$$

$$c_q'(\tilde{y}) = y_i^{-(d-q)} c_q(\tilde{\sigma}(\tilde{y}))$$

EFFECT ... IN CHART $\mathcal{U}_n' - \bigcup_{i\neq n} \mathcal{U}_i'$

in $\mathcal{U}_n'$ coordinates $y = (y_1, \dots, y_n)$

$$\sigma/\mathcal{U}_n' : \quad \begin{array}{ll} x_n = y_n & \\ x_j = y_n \cdot y_j & , j \in I \\ x = y & \notin I \cup \{n\} \end{array}$$

$$W := \mathcal{U}_n' - \bigcup_{i\in I} \mathcal{U}_i' = \{ y_i = 0 \quad \forall i \in I \}$$

$$\mathcal{U} \cap C = \{ x_n = 0 , \ x_j = 0 \quad \forall j \in I \}$$

$$(*) \qquad \boxed{\operatorname{ord}_C C_k \geq d-k > 0} \qquad k = 1, 2, \dots, d-2$$

$$f' = y_n^{-d} \cdot (f \circ \sigma) = \underbrace{(C_d \circ \sigma)(y)}_{\neq 0 \text{ on } W} + \sum_{0 \leq k < d} C_k'(y)$$

WHERE
$$\underbrace{C_k'(y)}_{= 0 \text{ on } W \text{ due to } (*)} := y_n^{-(d-k)} \cdot (C_k \circ \sigma)(y)$$

Hence, $\boxed{f' \neq 0 \quad \text{on } \mathcal{U}_n' - \bigcup_{i\in I} \mathcal{U}_i'}$

SUMMARY:

$C_d(x) \not\equiv 1$

$\tilde{x} = (x_1, \dots, x_{n-1})$ $\qquad d = \mathrm{ord}_0 f$

$$f = c_0(\tilde{x}) + \dots + c_{d-1}(\tilde{x})\, x_n^{d-1} + c_d(x) \cdot x_n^d$$

$$\frac{\partial^{d-1} f}{\partial x_n^{d-1}} \approx x_n \Rightarrow c_{d-1} \equiv 0 \ , \ c_d(x) \neq 0$$

$C = Z_I$

$\sigma_{|U_i'}$, $i \in I$: $\quad x_j = y_i \cdot y_j \quad j \in I \cup \{n\} - \{i\}$

$\qquad x_j = y_j \quad$ OTHER j

$$f'(y) = y_i^{-d} \cdot (f \circ \sigma)(y) \qquad \text{in } U_i'$$

$$f' = c_0'(\tilde{y}) + \dots + c_{d-2}'(\tilde{y})\, y_n^{d-2} + c_d'(y) \cdot y_n^d$$

$$c_q' := y_i^{-(d-q)} \cdot (c_q \circ \sigma)$$

THEN:

NEAR $\sigma^{-1}(C) = \{y_i = 0\}$

$$\frac{1}{d!} \frac{\partial^{d-1} f'}{\partial y_n^{d-1}} = y_n\, c_d'(y) + y_n^2 \cdot y_i \cdot (\dots) \approx y_n$$

$\boxed{\mathrm{ord}_y f' \leq d}$ AND $\boxed{\mathrm{ord}_y f' = d} \iff$

$\boxed{y_n = 0 \quad \text{AND} \quad \mathrm{ord}_y c_q' \geq d - q \ , q = 0, 1, \dots}$

NOTE: $\{y_n = 0\} = \{x_n = 0\}'$

PART 4. "WEAK" DESINGULARIZATION THEOREM. (PROOF IN ALL DETAILS.) WE WILL NEED THE FOLLOWING:

DEFINITION OF $\varphi \overset{def}{=}$ "$\psi \times id$" FOR "$\psi : N' \to N$" THAT OCCUR IN THE INDUCTIVE STEP OF PROOF BELOW:

ⓐ FOR A COORD. CHART $\mathcal{U}$ AND $N \overset{def}{=} \{x : g(x) = 0\}$ WITH $\frac{\partial g}{\partial x_n} \neq 0$ ON $\mathcal{U}$ AND A SMOOTH $C \overset{def}{=} \{x \in N : x_i = 0,\ i \in I\} =: Z_I$, $I \subset \{1, \dots, n-1\}$, AND $\psi \overset{def}{=} \sigma : N' \overset{def}{=} Bl_C N \to N$ A BLOW UP MAP

LET $\varphi \overset{def}{=}$ "$\psi \times id$" $: \mathcal{U}' \to \mathcal{U}$ DENOTE THE BLOW UP MAP $\mathcal{U}' \overset{def}{=} Bl_{\tilde{C}} \mathcal{U} \to \mathcal{U}$ WITH $\tilde{C} \overset{def}{=} \{x \in \mathcal{U} : x_i = 0,\ i \in I\}$.

ⓑ FOR A COORD. CHART $\mathcal{U}$ AND $N \overset{def}{=} \{x : g(x) = 0\}$ WITH $\frac{\partial g}{\partial x_n} \neq 0$ ON $\mathcal{U}$ AND FOR AN OPEN FINITE COVERING $N' = \bigcup_{I \ni i} N'_i$ WITH EACH $N'_i \overset{def}{=} N' - \{x_i = 0\}'$ AND $\psi \overset{def}{=} \coprod_{i \in I} N'_i \to \bigcup_{i \in I} N'_i$, A "COVERING" MAP, LET $\varphi \overset{def}{=}$ "$\psi \times id$" $M' \to M$, WHERE $M \overset{def}{=} \mathcal{U}'$, DENOTE THE "COVERING" MAP $M' \overset{def}{=} \coprod_{i \in I} \mathcal{U}'_i \longrightarrow \bigcup_{i \in I} \mathcal{U}'_i = \mathcal{U}' \overset{def}{=} Bl_{\tilde{C}} \mathcal{U}$, WHERE EACH $\mathcal{U}'_i := \mathcal{U}' - \{x_i = 0\}'$. NOTE: $\sigma|_{\mathcal{U}'_i} : \mathcal{U}'_i \to \mathcal{U}$ IN COORDINATES IS $\sigma : \begin{cases} x_j = y_i \cdot y_j & \text{FOR } j \in I - \{i\} \\ x_j = y_j & \text{OTHERWISE} \end{cases}$ AND $N'_i = \{x \in \mathcal{U}'_i : [y_i^{-1} (g \circ \sigma)(y)] = 0\} \hookrightarrow \mathcal{U}'_i$.

[I.H.E.S. (1988), see §4] — WITH BIERSTONE - [cf. J. of AMS §3 (1989)]

"WEAK" DESING. THM.: FOR $(f(x))$ ON SMOOTH M EXISTS $\varphi: M' \to M$ A COMPOSITE OF BLOW UPS (ALONG SMOOTH CENTERS) AND OF "COVERINGS" (i.e. maps $\coprod_j U_j \to \bigcup_j U_j$, OPEN U_j ...) SUCH THAT $(J_\varphi(x)) \cdot (f \circ \varphi)(x) = (x_1^{\Omega_1} \cdot \ldots \cdot x_n^{\Omega_n})$ locally (THE SO called N.CR.)

PROOF IN COMPLETE DETAILS BELOW.

DEFINITION: $[\mathrm{Sing}\, V(f)] \ni a$ IFF $(f(x)) = (g(x))^d$ "near a", $\mathrm{ord}_a g = 1$

$\mathrm{ord}_a f = d$

SET UP AND OTHER BASIC NOTIONS:

IN "YEAR" j $x \in M_j \xrightarrow{\sigma_j} \ldots \to M_{i+1} \xrightarrow{\sigma_{i+1}} M_i \to \ldots \to M_1 \to M_0 = M$

"ERA" OF HIGHER $d = \ldots$

E_j - EXC. HYPERSURFACES on M_j — ALL SMOOTH

$$(f \circ \varphi_j)(x) = \prod_{H \in E_j} \ell_H(x)^{m_H} \cdot f_0(x)$$

$$X_j \overset{\text{def}}{=} \{x : f_0(x) = 0\}, \quad \Sigma = \Sigma_j \overset{\text{def}}{=} \mathrm{Sing}\, X_j$$

$$d \overset{\text{def}}{=} \max\{\mathrm{ord}_x f_0 : x \in \Sigma\}$$

$S_d \overset{\text{def}}{=} \{x : \mathrm{ord}_x f_0 = d\}$ **PROP.** $a \in \Sigma \cap S_d \Rightarrow S_d \subseteq \Sigma$ NEAR a

$s(x) \overset{\text{def}}{=} \#\{H \in E_j : H \ni x$ AND "COMES" FROM "ERA" OF HIGHER $d\}$

PROOF OF PROP.: CHOOSE COORD. S.TH. $\frac{\partial^d f_0}{\partial x_n^d}(a) \neq 0$ THEN

$$N \overset{\text{def}}{=} \left\{x : \frac{\partial^{d-1} f_0}{\partial x_n^{d-1}}(x) = 0\right\} \supseteq S_d = \{x \in N : \mathrm{ord}_x C_k \geq d-k,\ 0 \leq k < d\}$$

WHERE $C_k \overset{\text{def}}{=} \left.\frac{\partial^k f_0}{\partial x_n^k}\right|_N$. FOR $x \in S_d$: $x \in \Sigma$ IFF SOME $C_k \neq 0$! DONE

$E'(a) \overset{\text{def}}{=} \{H \ni a : H$ "COMES" FROM "ERA" HIGHER $d\}$

PROOF OF "W. D." THM. INDUCTION ON $n = \dim M$ AND

(d, s) , $s \overset{\text{def}}{=} \max\{s(x) : x \in S_d \cap \Sigma\}$, **NAMELY:**

Ⓐ IN THE "YEAR" $i+1$ (WHEN d - JUST DECREASED) CHOOSE

$a \in S_d \cap \Sigma$ AND COORD. CHART: $\frac{\partial^d f_o}{\partial x_n^d} \neq 0$, $\frac{\partial \ell_H}{\partial x_n} \neq 0$, $\forall H \ni a$

$N \overset{\text{def}}{=} \{x : \frac{\partial^{d-1} f}{\partial x_n^{d-1}}(x) = 0\}$; $c_k \overset{\text{def}}{=} \frac{\partial^k f}{\partial x_n^k}\Big|_N$; $b_H \overset{\text{def}}{=} \ell_H\Big|_N$, $\forall H \in E'(a)$

$A \overset{\text{def}}{=} \prod$ "all" $c_k^{d!/d-k}$ · "all" $b_H^{d!}$ · "all" their differences $\neq 0$

$A \in \mathcal{O}(N)$, $\dim N = n-1 < n \Rightarrow \exists\, \psi : N' \to N \ldots (A \circ \psi)$ N.CR.

$\boxed{\varphi \overset{\text{def}}{=} \text{"}\psi \times id\text{"} : M' \to M}$ "$a \longmapsto a_o \overset{\text{def}}{=} a$" THEN

LEMMA 1 ✱ $\begin{cases} \text{EACH } c_k^{d!/d-k} \approx \tilde{x}^{\Omega(k)} , \quad b_H^{d!} \approx \tilde{x}^{\Omega_H} \quad \tilde{x} = (x_1, \ldots, x_{n-1}) \\ \mathcal{E}xp \overset{\text{def}}{=} \{\ldots \Omega(k) \;;\; \ldots \Omega_H\} \subseteq \mathbb{N}^{n-1} \textbf{ is totally ordered!} \end{cases}$

PROOF. DIRECT AND EASY.

$\Longrightarrow \exists \min \mathcal{E}xp =: \Omega \quad \Longrightarrow \quad S_d \cap \left[\bigcap_{H \in E'(a)} H\right] =$

Ⓑ $= \{x \in N : \mathrm{ord}_x c_k \geq d-k \;\; \forall k ,\; \mathrm{ord}_x b_H \geq 1 \;\; \forall H \in E'(a)\} =$

$= \{x \in N : \mathrm{ord}\, \tilde{x}^{\Omega} \geq d!\} = \bigcup_I Z_I$ **EACH** $Z_I \overset{\text{def}}{=}$

$= \{x \in N : x_i = 0 \;\; \forall i \in I\}$ AND **I minimal** $\subseteq \{1, \ldots, n-1\}$

SUCH THAT $\sum_{i \in I} \Omega_i \geq d! \Leftrightarrow 0 \leq \sum_{i \in I} \Omega_i - d! < \Omega_j \quad \forall j \in I$.

LEMMA 2

CONSIDER BLOW UP WITH $C = Z_I$ (any): $\mathcal{U}' \xrightarrow{\sigma} \mathcal{U}$

$\mathcal{U}' = \bigcup_{j \in I \cup \{n\}} \mathcal{U}'_j$, $\mathcal{U}'_n \overset{\text{def}}{=} \mathcal{U}' - N'$ THEN $\forall x \in \mathcal{U}'$, $C \subseteq S_d \Rightarrow \mathrm{ord}_x f'_o \leq d$

IF $\mathrm{ord}_\alpha f'_o = d \Rightarrow \alpha \in N' \cap \mathcal{U}'_j$, some $j \in I$ $\quad \sigma|_{\mathcal{U}'_j} : \begin{cases} x_i = y_j \cdot y_i & i \in I \setminus n\text{-}j \\ x_i = y_i & \text{OTHERWISE} \end{cases}$

PROOF. FOLLOWS FROM "THE EFFECT OF EQUIMULTIPLE

BLOW UP" CALCULATION.

Also, (THE "EFFECT OF EQUIMULTIPLE BLOW UP" CALCULATION), $C'_k = y_j^{-(d-k)} \cdot (C_k \circ \sigma)$, $b'_H = y_j^{-1} \cdot (b_H \circ \sigma)$.
HENCE,
✱ AGAIN IS VALID SINCE $y_j^{-d!} \cdot (\tilde{x}^{\Omega} \circ \sigma) = \tilde{y}^{\Omega'}$ WITH $\Omega'_i = \Omega_i$ FOR $i \neq j$ AND $\Omega'_j = \sum_{i \in I} \Omega_i - d!$ $(\forall \Omega \in \mathbb{N}^{n-1})$.

FOR $\Omega = \min[\mathcal{E}xp \text{ AT } a]$ AS ABOVE $\therefore$ $\Omega'_j < \Omega_j$ $(j \in I)$ AND $\Omega' = \min[\mathcal{E}xp \text{ AT } a']$. BUT $0 \leq \sum_i \Omega'_i < \sum_i \Omega_i$ (AND THESE ARE FROM $\mathbb{N}$) AND IF $s(a') = s$ THEN $\sum_i \Omega'_i \geq d$:
WE CONTINUE "THESE" BLOW UPS (UNLESS THE RESPECTIVE $\Sigma_j = \emptyset$) UNTIL (d, s) decreases (NEED NO MORE THEN $\sum_i \Omega_i$ BLOW UPS, WHERE Ω is AS IN ✱ AT a). IF d decreases WE START WITH INDUCTION ON $n = \dim M$ STEP ALL OVER...

© OTHERWISE (i.e. EITHER $\Sigma_j = \emptyset$ OR s IS SMALLER, $\Sigma_j \neq \emptyset$) WE CONTINUE THESE BLOW UPS (WHICH WE MAY SINCE ALTHOUGH FOR SOME $H \in E^1(a)$, $H' \not\ni a'$, i.e. $\Omega_{H'} = 0$, ✱ REMAINS VALID WITH $H' \in E^1(a')$, i.e. SUCH THAT $\Omega_{H'} \neq 0$, AND "NEW" $\Omega = \min[\mathcal{E}xp \text{ AT } a'] \neq 0$) UNTIL EITHER d-decreases OR (IF $\Sigma_j = \emptyset$) $s(x) \leq 1$ $\forall x$ AND $s(x) = 0$ $\forall x \in X_j$. **DONE. END OF PROOF OF THM**